

THE 1,2-COLOURED HOMFLY-PT LINK HOMOLOGY

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ABSTRACT. In this paper we define the 1,2-coloured HOMFLY-PT triply graded link homology and prove that it is a link invariant. We also conjecture on how to generalize our construction for arbitrary colours.

1. INTRODUCTION

In this paper we define the coloured HOMFLY-PT triply graded link homology for links whose components are labelled 1 or 2. The point of this paper is to generalize Khovanov's [6] construction of the HOMFLY-PT link homology, originally due to Khovanov and Rozansky [8]. The Euler characteristic of our triply graded link homology is a 2-variable polynomial which generalizes the HOMFLY-PT polynomial. We did not develop a complete recursive combinatorial calculus for it, as in [5] and [9]. However, in Section 2 we will indicate some relations which should hold in such a calculus and which are closely related to the ones in [9].

The 1,2-coloured HOMFLY-PT link homology is a triply graded link homology, just as the ordinary HOMFLY-PT link homology due to Khovanov and Rozansky [8]. In [4] such a link homology was conjectured to exist from the physics point of view. We follow the approach using bimodules and Hochschild homology, as was done for the ordinary HOMFLY-PT link homology by Khovanov in [6]. With Rasmussen's results for the ordinary HOMFLY-PT homology in mind, we conjecture that, in a certain sense, our link homology is the limit of as yet undefined 1,2-coloured $sl(N)$ link homologies, when N goes to infinity. Note that the graded Euler characteristic of these 1,2-coloured $sl(N)$ link homologies will be the Reshetikhin-Turaev polynomials for these colorings.

Although these $sl(N)$ link homologies have not yet been defined, there has been some progress made towards their definition in [15, 16]. In those papers the matrix factorization approach is followed. One should also be able to define the 1,2-coloured HOMFLY-PT link homology using matrix factorizations, and in some sense this should be equivalent to our approach. For technical reasons the bimodule approach is slightly easier, which is why we have not used matrix factorizations. As for the

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ordinary HOMFLY-PT link homology, we use braid presentations of the links and prove invariance under the braidlike Reidemeister moves and the Markov moves.

In the last section of this paper we have sketched how to define the coloured HOMFLY-PT link homology for arbitrary colours and how to prove its invariance. The underlying ideas are the same, but the actual calculations are much harder. One needs a different technique to handle those calculations for arbitrary colours. In the meantime, Webster and Williamson [14] have defined the general coloured HOMFLY-PT link homology using geometric techniques and have confirmed our conjectures. In a future paper we hope to give algebraic proofs of our conjectures.

To compute the 1,2-coloured HOMFLY-PT link homology is very hard. In [4] there is a conjecture for the Hopf link, which we confirm by our calculations in Section 7, but we have not done more complicated calculations. One road to follow would be the one Rasmussen showed for the ordinary HOMFLY-PT link homology [10]. Once the $sl(N)$ -link homologies have been defined, there should be spectral sequences from the coloured HOMFLY-PT link homologies to the $sl(N)$ -link homologies. For “small” knots these spectral sequences should collapse for low values of N , which might make them computable. Another road to follow would be the one initiated by Webster and Williamson [14]. This approach might give some results for certain classes of knots, such as the torus knots.

Finally, let us briefly sketch an outline of our paper. In Section 2 we recall some basic facts from [9] about the combinatorial calculus of the 1,2-coloured HOMFLY-PT polynomial and define one version of part of it that we shall categorify. In Section 3 we categorify this part of the calculus by using bimodules. In Section 4 we define the 1,2-coloured HOMFLY-PT link homology. As stated before, we use a braid presentation of the link. In Section 5 we prove invariance of the 1,2-coloured HOMFLY-PT link homology under the braidlike Reidemeister moves II and III. In Section 6 we prove its invariance under the Markov moves. In Section 7 we have included the calculation of the 1,2-coloured HOMFLY-PT homology of the Hopf link. In Section 8 we sketch the definition of the coloured HOMFLY-PT link homology for arbitrary colours and conjecture its invariance under the second and third Reidemeister moves and the Markov moves.

Although we have tried to write a fairly self-contained paper, some familiarity with [6, 7, 8, 5, 9] will probably help the reader in understanding this paper.

2. THE MOY CALCULUS

In this section we explain part of the MOY calculus for the 1,2-coloured HOMFLY-PT link polynomial. The reader might want to compare our relations to the analogous ones in [9]. As already remarked in the introduction this part of the calculus probably does not give a complete recursive calculus for the polynomial invariant. At least we do not know any proof of such a fact. We have simply picked those relations that are necessary for proving the invariance of the polynomial. As it turns out, their categorifications prove the invariance of the related link homology. Before we go on, let us remark that this section is merely motivational. It is meant to help the reader understand why we set up things as we did. No part of the construction of the link homology or the proof of its invariance depends on this section. As a matter of fact, we could have put this section at the end of our paper as a consequence of the construction of the link homology, i.e. we can obtain the

1,2-coloured HOMFLY-PT polynomial as the graded Euler characteristic of our link homology.

The calculus uses labelled trivalent graphs, which we call MOY webs, and is similar to the MOY calculus from [9] and generalizes the calculus used by Khovanov and Rozansky in [8] to define triply graded link homology. The resolutions of link diagrams consist of MOY webs whose edges are labelled by positive integers, such that at each trivalent vertex the sum of the labels of the outgoing edges equals the sum of the labels of ingoing edges. Although the theory can be extended to allow for general labellings, in this paper we shall only consider the ones where the labellings of the edges are from the set $\{1, 2, 3, 4\}$.

First we introduce the calculus for such graphs. This is an extension of the one with labellings being only 1 and 2 (see [8]), and a variant of the one from [9]. We require the following axioms to hold:

$$(A1) \quad \text{circle with } k \text{ arrows} = \prod_{i=1}^k \frac{1 + t^{-1}q^{2i-1}}{1 - q^{2i}}$$

$$(A2) \quad \text{trivalent vertex with } i, j \text{ on left and } i \text{ on right} = \prod_{l=1}^j \frac{1 + t^{-1}q^{2i+2l-1}}{1 - q^{2l}}$$

$$(A3) \quad \text{trivalent vertex with } i, j \text{ on left and } i+j \text{ on right} = \begin{bmatrix} i+j \\ i \end{bmatrix} \uparrow_{i+j}$$

$$(A4) \quad \text{trivalent vertex with } i, j, k \text{ on left and } i+j+k \text{ on right} = \text{trivalent vertex with } i, j, k \text{ on left and } i+j \text{ on right}$$

$$(A5) \quad \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 2 \text{ on right} = \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 3 \text{ on right} + q^2 \uparrow_{2,1}$$

$$(A6) \quad \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 2 \text{ on right} = \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 4 \text{ on right} + (q^2 + q^4) \uparrow_{3,1}$$

$$(A7) \quad \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 2 \text{ on right} = \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 4 \text{ on right} + q^2 \text{trivalent vertex with } 1, 1, 1 \text{ on left and } 2 \text{ on right}$$

In this paper we use the following (non-standard) convention for the quantum integers:

$$[n] = 1 + q^2 + \dots + q^{2(n-1)} = \frac{1 - q^{2n}}{1 - q^2}$$

We define the quantum factorial and the binomial coefficients in the standard way:

$$[n]! = [1][2] \cdots [n], \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}.$$

Although we have defined the axioms (A1)-(A3) for arbitrary i, j and k , in this paper we shall only need them in the cases where the indices i, j and k are from the set $\{1, 2\}$.

Since this section is only motivational, we also include a sketch of the 2-variable 1,2-coloured HOMFLY-PT link polynomial based on the above diagrammatic calculus, although the definition is necessarily incomplete. The polynomial would be defined for link projections which have the form of the closure of a braid. All positive and negative crossings between strands labelled 2 would be resolved in three different resolutions. The bracket would be defined using the following relations:

(2.1)

The resolutions of the positive and negative crossings between strands labelled 1 and 2 would be given by

(2.2)

One would obtain the resolutions, when the labels 1 and 2 are swapped, by rotation around the y-axis.

Finally, the case of the crossings when both strands are labelled 1 would be the same as in [8]:

(2.3)

Assuming that axioms (A1)-(A7) could be extended such that any MOY web can be evaluated, we could define a polynomial $\langle D \rangle$ for each link diagram D with components labelled by 1 and 2, using the resolutions in (2.1), (2.2) and (2.3). Analogously as in [9], it can be shown that it would be invariant under the second and the third Reidemeister moves and would have the following simple behaviour under the first Reidemeister move:

$$\begin{array}{c} \uparrow \\ \text{loop} \\ \downarrow \\ 2 \end{array} = \begin{array}{c} \uparrow \\ \downarrow \\ 2 \end{array}, \quad \begin{array}{c} \uparrow \\ \text{loop} \\ \downarrow \\ 2 \end{array} = t^{-2}q^{-2} \begin{array}{c} \uparrow \\ \downarrow \\ 2 \end{array}$$

when the strand is labelled 2 and

$$\begin{array}{c} \uparrow \\ \text{loop} \\ \downarrow \\ 1 \end{array} = \begin{array}{c} \uparrow \\ \downarrow \\ 1 \end{array}, \quad \begin{array}{c} \uparrow \\ \text{loop} \\ \downarrow \\ 1 \end{array} = -t^{-1}q^{-1} \begin{array}{c} \uparrow \\ \downarrow \\ 1 \end{array}$$

when the strand is labelled 1.

To obtain a genuine knot invariant, we would therefore have to multiply the bracket by the following overall factor:

$$I(D) = (-tq)^{\frac{-n_+^1+n_-^1+s_1(D)-2n_+^2+2n_-^2+2s_2(D)}{2}} \langle D \rangle,$$

where n_+^i and n_-^i denote the number of positive and negative crossings, respectively, between two strands labelled i and where $s_i(D)$ denotes the number of strands labelled i for $i = 1, 2$.

3. THE CATEGORIFICATION OF THE MOY CALCULUS

In this section we show which bimodule to associate to a web and that these bimodules satisfy axioms (A3)-(A7) up to isomorphism. The proof that (A1) and (A2) are also satisfied will be given in Section 6 after we have explained the Hochschild homology. We only explain the general idea and work out the bits which involve edges with higher labels and which have not been explained by Khovanov in [6].

Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the ring of complex polynomials in n variables. We introduce a grading on R by defining $\deg x_i = 2$ for every $i = 1, \dots, n$. This grading is called the q -grading. For any partition i_1, \dots, i_k of n , let $R_{i_1 \dots i_k}$ denote the subring of R of complex polynomials which are invariant under the product of the symmetric groups $S_{i_1} \times \dots \times S_{i_k}$. For starters we associate a bimodule to each MOY-web (see Section 2). Suppose Γ is a MOY web with k bottom ends labelled by i_1, \dots, i_k and m top ends labelled by j_1, \dots, j_m . Recall that $i_1 + \dots + i_k = j_1 + \dots + j_m$ holds and let R have exactly that number of variables. We associate an $R_{j_1 \dots j_m} - R_{i_1 \dots i_k}$ -bimodule to Γ . We read Γ from bottom to top. To the bottom edges we associate the bimodule $R_{i_1 \dots i_k}$. When we move up in the web we encounter a Υ -shaped or a \wedge -shaped bifurcation. The Υ -shape will always correspond to induction and the \wedge to restriction; e.g. if we first encounter a Υ -shaped bifurcation which splits i_1 into i_1^0 and i_1^1 , then we tensor $R_{i_1 \dots i_k}$ on the left with $R_{i_1^0 i_1^1 \dots i_k}$ over $R_{i_1 \dots i_k}$. We will write this tensor product as

$$R_{i_1^0 i_1^1 \dots i_k} \otimes_{i_1 \dots i_k} R_{i_1 \dots i_k}.$$

If we first encounter a \wedge -shaped bifurcation with joints i_1 and i_2 , then we restrict the left action on $R_{i_1 \dots i_k}$ to $R_{i_1+i_2 \dots i_k}$. Note that the latter action goes unnoticed

until tensoring again at some next Υ -shaped bifurcation, in which case we tensor over the smaller ring, or until taking the Hochschild homology in a later stage. As we go up in the web we use either induction or restriction at each bifurcation. This way an $R_{j_1 \dots j_m} - R_{i_1 \dots i_k}$ -bimodule associated to Γ is obtained, which we denote by $\widehat{\Gamma}$. Note that if we have two MOY-webs, Γ and Γ' , such that the bottom of Γ' (labelled by i_1, \dots, i_k) can be glued onto the top of Γ , then

$$\widehat{\Gamma\Gamma'} \cong \widehat{\Gamma'} \otimes_{R_{i_1 \dots i_k}} \widehat{\Gamma}.$$

The identity web in Figure 1 whose edges are labelled by i_1, \dots, i_k will always

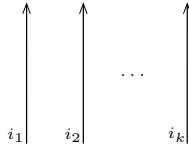


FIGURE 1. The identity web $\uparrow_{i_1 \dots i_k}$

be denoted by $\uparrow_{i_1 \dots i_k}$ and $\widehat{\uparrow}_{i_1 \dots i_k} = R_{i_1 \dots i_k}$. The dumbbell web in Figure 2 whose

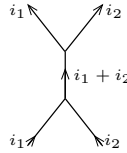


FIGURE 2. The dumbbell web $\Upsilon_{i_1 i_2}$

outer edges are labelled i_1 and i_2 will always be denoted by $\Upsilon_{i_1 i_2}$ and $\widehat{\Upsilon}_{i_1 i_2} = R_{i_1 i_2} \otimes_{i_1 + i_2} R_{i_1 i_2}$. Since the bimodules that we use are graded, we can apply a grading shift. In the text and when using small symbols we denote a positive shift of k values applied to a bimodule M by $M\{k\}$. When we use MOY-type pictures we denote the same shift by q^k . Note that we also do not put a hat on top of these pictures to avoid too much notation and unnecessarily large figures.

Note that for our construction we first need to choose a height function on the web. However, it is easy to see that the bimodule does not depend on the choice of this height function.

Now that we know which bimodule to associate to a MOY web, we will show some direct sum decompositions for bimodules associated to certain MOY webs. These decompositions are necessary to show that our bimodules indeed categorify the MOY calculus and also to show that the link homology is invariant, up to isomorphism, under the Reidemeister moves.

Lemma 3.1. *Let δ_{ij} be the digon in (A3). Then we have*

$$\widehat{\delta}_{ij} \cong \begin{bmatrix} i+j \\ i \end{bmatrix} \widehat{\uparrow}_{i+j}.$$

Note that the quantum binomial can be written as a sum of powers of q . By

$$\begin{bmatrix} i+j \\ i \end{bmatrix} \widehat{\uparrow}_{i+j}$$

we mean the corresponding direct sum of copies of $\hat{\uparrow}_{i+j}$, where each copy is shifted by the correct power of q .

Proof. Note that R_{ij} as an R_{i+j} -(bi)module is isomorphic to $H_{U(i+j)}^*(G(i, i+j))$, the $U(i+j)$ equivariant cohomology of the complex Grassmannian $G(i, i+j)$. Therefore the Schur polynomials $\pi_{k_1 \dots k_i}$ in the first i variables, for $0 \leq k_i \leq \dots \leq k_1 \leq j$, form a basis of R_{ij} as an R_{i+j} -(bi)module (see [3], for example). Alternatively we can use $G(j, i+j)$ and obtain that the Schur polynomials $\pi'_{\ell_1 \dots \ell_j}$ in the last j variables form a basis of R_{ij} as an R_{i+j} -(bi)module for $0 \leq \ell_j \leq \dots \leq \ell_1 \leq i$. This shows that

$$R_{ij} \cong \begin{bmatrix} i+j \\ i \end{bmatrix} R_{i+j}$$

holds. The proof of this lemma follows, since $\hat{\phi}_{ij} = R_{ij} \otimes_{i+j} R_{i+j}$ and $\hat{\uparrow}_{i+j} = R_{i+j}$. \square

Before we continue with square decompositions we define the following bimodule maps:

Definition 3.2. We define the R_{1k} -bimodule maps

$$\mu_{1k} : \widehat{\mathfrak{X}}_{1k} \rightarrow \widehat{\uparrow}_{1k} \quad \text{and} \quad \Delta_{1k} : \widehat{\uparrow}_{1k} \rightarrow \widehat{\mathfrak{X}}_{1k}$$

by

$$\mu_{1k}(a \otimes b) = ab \quad \text{and} \quad \Delta_{1k}(1) = \sum_{j=0}^k (-1)^j e_j \left(\sum_{i=0}^{k-j} x_1^{k-j-i} \otimes x_1^i \right).$$

The elements e_i are the elementary symmetric polynomials in $k+1$ variables.

The formula for Δ_{1k} can also be written as

$$\Delta_{1k}(1) = \sum_{j=0}^k (-1)^j x_1^{k-j} \otimes e'_j,$$

where e'_j is the j -th elementary symmetric polynomial in the last k variables x_2, \dots, x_{k+1} . Note that μ_{1k} has degree 0 and Δ_{1k} has degree $2k$.

It is not hard to see that μ and Δ are indeed R_{1k} bimodule maps. One can check this by direct computation or, as above, note that R_{1k} is isomorphic to $H_{U(k+1)}^*(G(1, k))$ as an R_{k+1} -module. The maps above are well known (it is an immediate consequence of exercise 1.1 of lecture 4 in [3], for example) to be the multiplication and comultiplication in this commutative Frobenius extension with respect to the trace defined by $\text{tr}(x_1^k) = 1$. The observation now follows from the fact that the multiplication and comultiplication in a commutative Frobenius extension A are always A -bimodule maps.

When it is not immediately clear to which variables one applies a multiplication or comultiplication, we will indicate them in a superscript. When there is no confusion possible, we will write ab for $\mu_{1,k}(a \otimes b)$.

Definition 3.3. We define the R_{22} -bimodule maps

$$\mu_{22} : \widehat{\mathfrak{X}}_{22} \rightarrow \widehat{\uparrow}_{22} \quad \text{and} \quad \Delta_{22} : \widehat{\uparrow}_{22} \rightarrow \widehat{\mathfrak{X}}_{22}$$

by

$$\mu_{22}(a \otimes b) = ab$$

and

$$\Delta_{22}(1) = \pi_{22} \otimes 1 - \pi_{21} \cdot \pi'_{10} + \pi_{20} \otimes \pi'_{11} + \pi_{11} \cdot \pi'_{20} - \pi_{10} \cdot \pi'_{21} + 1 \cdot \pi'_{22} + \leftrightarrow .$$

The π_{ij} and π'_{ij} are the Schur polynomials in x_1, x_2 and x_3, x_4 , respectively. The \leftrightarrow indicates the terms which are obtained from all the previous terms by interchanging the π_{ij} and the π'_{ij} so that Δ becomes cocommutative.

One easily checks that μ_{22} and Δ_{22} are R_{22} -bimodule maps by direct computation. They have degree 0 and 8, respectively. The map Δ_{22} is the comultiplication in $H_{U(4)}^*(G(2, 4))$ with respect to the trace defined by $\text{tr}(\pi_{22}) = 1$. Sometimes it will be useful to rewrite the formula of Δ_{22} entirely in terms of the π'_{ij} and elements of R_4 . To do so, use the following relations:

$$\begin{aligned} \pi_{10} &= \pi_{1000} - \pi'_{10}, \\ \pi_{11} &= \pi_{1100} - \pi'_{10}\pi_{1000} + \pi'_{20}, \\ \pi_{20} &= \pi_{2000} - \pi'_{10}e_1 + \pi'_{11}, \\ \pi_{21} &= \pi_{2100} - \pi'_{10}(\pi_{2000} + \pi_{1100}) + \pi'_{20}\pi_{1000} + \pi'_{11}\pi_{1000} - \pi'_{21}, \\ \pi_{22} &= \pi_{2200} - \pi'_{10}\pi_{2100} + \pi'_{11}\pi_{2000} + \pi'_{20}\pi_{1100} - \pi'_{21}\pi_{1000} + \pi'_{22}. \end{aligned}$$

We can now prove some square decompositions.

Lemma 3.4. *Let \mathfrak{H}_{1112} be the square in (A5). The following decomposition holds:*

$$\widehat{\mathfrak{H}}_{1112} \cong \widehat{\mathfrak{X}}_{21} \oplus \widehat{\uparrow}_{21}\{2\}.$$

Proof. Note that

$$\widehat{\mathfrak{H}}_{1112} = R_{111} \otimes_{12} R_{111} \otimes_{21} R_{21} \quad \text{and} \quad \widehat{\mathfrak{X}}_{21} = R_{21} \otimes_3 R_{21}.$$

Let Γ_i be the webs in Figure 3, for $i = 1, 2$. Then we have

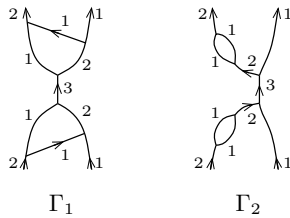


FIGURE 3. Intermediate webs Γ_1 and Γ_2

$$\widehat{\Gamma}_1 = R_{111} \otimes_{12} R_{12} \otimes_3 R_{111} \otimes_{21} R_{21} \cong R_{111} \otimes_{21} R_{21} \otimes_3 R_{111} \otimes_{21} R_{21} \cong \widehat{\Gamma}_2.$$

The map $f: \widehat{\mathfrak{H}}_{1112} \rightarrow \widehat{\mathfrak{X}}_{21}$ is the composite of

$$\widehat{\mathfrak{H}}_{1112} \xrightarrow{f_1} \widehat{\Gamma}_1\{-4\} \cong \widehat{\Gamma}_2\{-4\} \xrightarrow{f_2} \widehat{\mathfrak{X}}_{21}.$$

We define f_1 by

$$f_1(a \otimes b \otimes c) = a \otimes \Delta_{12}(b) \otimes c,$$

where Δ_{12} is defined as in Definition 3.2. The map f_2 is defined by applying to both digons the map $R_{111} \rightarrow R_{21}\{2\}$ which corresponds to the projection onto the second summand in the decomposition $R_{111} \cong R_{21} \oplus x_1 R_{21}$; e.g. we have

$$x_1 \mapsto 1 \quad \text{and} \quad x_2 = (x_1 + x_2) - x_1 \mapsto -1 \quad \text{and} \quad x_1^2 = (x_1 + x_2)x_1 - x_1x_2 \mapsto x_1 + x_2.$$

Similarly define $g: \widehat{\chi}_{21} \rightarrow \widehat{\mathfrak{H}}_{1112}$ as the composite of

$$\widehat{\chi}_{21} \xrightarrow{g_1} \widehat{\Gamma}_2 \cong \widehat{\Gamma}_1 \xrightarrow{g_2} \widehat{\mathfrak{H}}_{1112}.$$

We define g_1 by twice applying the inclusion map $R_{21} \hookrightarrow R_{111}$ to create the digons. The map g_2 is defined by

$$g_2(a \otimes b \otimes c \otimes d) = a \otimes b c \otimes d,$$

where c is mapped to R_{12} by the inclusion map before applying μ_{12} . One easily verifies by direct computation that $fg = \text{id}$, so $\widehat{\chi}_{21}$ is a direct summand of $\widehat{\mathfrak{H}}_{1112}$.

To show that $\widehat{\uparrow}_{21}$ is a direct summand as well, we also use an intermediate web, denoted $\widehat{\Gamma}_3$ and shown in Figure 4. Note that $\widehat{\Gamma}_3 = R_{111} \otimes_{21} R_{21}$. Define

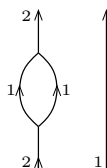


FIGURE 4. Intermediate web Γ_3

$h: \widehat{\mathfrak{H}}_{1112} \rightarrow \widehat{\uparrow}_{21}$ as the composite of

$$\widehat{\mathfrak{H}}_{1112} \xrightarrow{h_1} \widehat{\Gamma}_3 \xrightarrow{h_2} \widehat{\uparrow}_{21}\{2\}.$$

In this case h_1 is given by

$$h_1(a \otimes b \otimes c) = a b \otimes c$$

and h_2 by applying the same projection $R_{111} \rightarrow R_{21}\{2\}$ as above. Inversely, we define $j: \widehat{\uparrow}_{21} \rightarrow \widehat{\mathfrak{H}}_{1112}$ as the composite of

$$\widehat{\uparrow}_{21}\{2\} \xrightarrow{j_1} \widehat{\Gamma}_3\{2\} \xrightarrow{j_2} \widehat{\mathfrak{H}}_{1112}.$$

The first map is defined by the inclusion $R_{21} \hookrightarrow R_{111}$. The second map is defined by

$$j_2(a \otimes b) = \Delta_{11}^{x_2, x_3}(a) \otimes b.$$

Again, by direct computation, it is straightforward to check that $hj = -2\text{id}$, so $\widehat{\uparrow}_{21}\{2\}$ is also a direct summand of $\widehat{\mathfrak{H}}_{1112}$.

Also by direct computation one easily checks that $hg = 0$ and $fj = 0$. Finally we have to show that (g, j) is surjective. Since all maps involved are bimodule maps and $R_{111} \cong R_{21} \oplus x_2 R_{21}$ and $R_{111} \cong R_{12} \oplus x_3 R_{12}$, we only have to show that $1 \otimes 1 \otimes 1$ and $1 \otimes x_2 \otimes 1$ are in its image. We have $g(1 \otimes 1) = 1 \otimes 1 \otimes 1$ and $j(1) + g(x_3 \otimes 1) = 1 \otimes x_2 \otimes 1$. This finishes the proof of the lemma. \square

Lemma 3.5. *Let $\widehat{\mathfrak{H}}_{1112}$ be the square in (A6). Then we have*

$$\widehat{\mathfrak{H}}_{1112} \cong \widehat{\chi}_{31} \oplus \widehat{\uparrow}_{31}\{2\} \oplus \widehat{\uparrow}_{31}\{4\}.$$

Proof. The arguments are analogous to the ones used in the proof of Lemma 3.4. To show that $\widehat{\chi}_{22}$ is a direct summand of $\widehat{\mathfrak{H}}_{1112}$, use the intermediate webs in Figure 5.

With the same notation as before, let

$$f_1(a \otimes b \otimes c) = a \otimes \Delta_{22}(b) \otimes c,$$

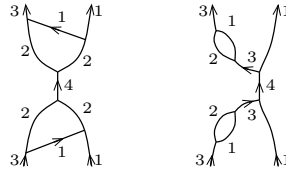


FIGURE 5. Intermediate webs for \mathfrak{H}_{1122}

and for f_2 use for both digons the map $R_{211} \rightarrow R_{31}\{4\}$ which corresponds to the projection onto the third direct summand in the decomposition $R_{211} \cong R_{31} \oplus x_3 R_{31} \oplus x_3^2 R_{31}$.

For g_1 we use the inclusion $R_{31} \hookrightarrow R_{211}$ twice to create the digons and

$$g_2(a \otimes b \otimes c \otimes d) = a \otimes b c \otimes d.$$

To show that $\widehat{\uparrow}_{22}\{2\} \oplus \widehat{\uparrow}_{22}\{4\}$ is a direct summand of $\widehat{\mathfrak{H}}_{1122}$, use the intermediate web in Figure 6.

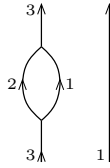


FIGURE 6. Intermediate web

Define h_1 by

$$h_1(a \otimes b \otimes c) = a b \otimes c$$

and h_2 by applying the map $R_{211} \rightarrow R_{31}\{2\} \oplus R_{31}\{4\}$ corresponding to the projection on the last two direct summands in the decomposition of R_{211} above.

For j_1 use the map $R_{31}\{2\} \oplus R_{31}\{4\} \rightarrow R_{211}$ defined by

$$(1, 0) \mapsto 1 \quad \text{and} \quad (0, 1) \mapsto x_3$$

to create the digon. Define j_2 by

$$j_2(a \otimes b) = \Delta_{11}^{x_3, x_4}(a) \otimes b.$$

An easy calculation shows that

$$h j = \begin{pmatrix} 1 & -x_4 \\ 0 & 1 \end{pmatrix},$$

which is invertible. This shows that $R_{31}\{2\} \oplus R_{31}\{4\}$ is a direct summand of $\widehat{\mathfrak{H}}_{1122}$.

Using the rewriting rules for Δ_{22} , which were given below its definition, one easily checks that $g f = 2 \text{id}$. Therefore $\widehat{\mathfrak{X}}_{31}$ is a direct summand of $\widehat{\mathfrak{H}}_{1122}$, too.

Another easy calculation shows that $h g = 0$ and a slightly harder one that $f j = 0$.

It remains to show that (g, j) is surjective. It suffices to show that $1 \otimes 1 \otimes 1$, $1 \otimes x_3 \otimes 1$ and $1 \otimes x_3^2 \otimes 1$ are in its image. We have

$$\begin{aligned} 1 \otimes 1 \otimes 1 &= g(1 \otimes 1), \\ 1 \otimes x_3 \otimes 1 &= j(1, 0) + g(x_4 \otimes 1), \\ 1 \otimes x_3^2 \otimes 1 &= j(0, 1) + j(x_4, 0) + g(x_4^2 \otimes 1). \end{aligned} \quad \square$$

Lemma 3.6. *Let $\widehat{\mathfrak{H}}_{2113}$ be the square in (A7). We have*

$$\widehat{\mathfrak{H}}_{2113} \cong \widehat{\mathfrak{X}}_{22}^{13} \oplus \widehat{\mathfrak{Y}}_{22}^{13}\{2\}.$$

Proof. We first define the bimodule map

$$\phi_1: \widehat{\mathfrak{X}}_{22}^{13} = R_{13} \otimes_4 R_{22} \rightarrow R_{121} \otimes_{31} R_{211} \otimes_{22} R_{22} = \widehat{\mathfrak{H}}_{2113}.$$

We use the two intermediate bimodules $\widehat{\Gamma}_1 = R_{121} \otimes_{13} R_{13} \otimes_4 R_{211} \otimes_{22} R_{22}$ and $\widehat{\Gamma}_2 = R_{121} \otimes_{31} R_{31} \otimes_4 R_{211} \otimes_{22} R_{22}$ (see Figure 7). Then ϕ_1 is the composite

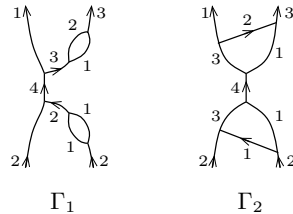


FIGURE 7. Intermediate webs Γ_1 and Γ_2

$$\widehat{\mathfrak{X}}_{22}^{13} \xrightarrow{\phi_1^1} \widehat{\Gamma}_1 \xrightarrow{\phi_1^2} \widehat{\Gamma}_2 \xrightarrow{\phi_1^3} \widehat{\mathfrak{H}}_{2113}$$

with

$$\begin{aligned} \phi_1^1(a \otimes b) &= 1 \otimes a \otimes 1 \otimes b, \\ \phi_1^2(a \otimes b \otimes c \otimes d) &= ab \otimes 1 \otimes c \otimes d, \\ \phi_1^3(a \otimes b \otimes c \otimes d) &= a \otimes bc \otimes d. \end{aligned}$$

Conversely, we define ψ_1 as the composite

$$\widehat{\mathfrak{H}}_{2113} \xrightarrow{\psi_1^1} \widehat{\Gamma}_2 \xrightarrow{\psi_1^2} \widehat{\Gamma}_1 \xrightarrow{\psi_1^3} \widehat{\mathfrak{X}}_{22}^{13}$$

with

$$\begin{aligned} \psi_1^1(a \otimes b \otimes c) &= a \otimes \Delta_{31}(b) \otimes c, \\ \psi_1^2(a \otimes b \otimes c \otimes d) &= ab \otimes 1 \otimes c \otimes d \end{aligned}$$

and with ψ_1^3 defined by applying the maps $R_{121} \rightarrow R_{13}\{4\}$ and $R_{211} \rightarrow R_{22}\{2\}$ which are the projections onto the last direct summands in the decompositions $R_{121} \cong R_{13} \oplus x_4 R_{13} \oplus x_4^2 R_{13}$ and $R_{211} \cong R_{22} \oplus x_3 R_{22}$. A short calculation shows that $\psi_1 \phi_1 = -\text{id}$, which proves that $\widehat{\mathfrak{X}}_{22}^{13}$ is a direct summand of $\widehat{\mathfrak{H}}_{2113}$.

Let us now define $\phi_2: \widehat{\mathfrak{Y}}_{22}^{13} \rightarrow \widehat{\mathfrak{H}}_{2113}$. Note that $\widehat{\mathfrak{Y}}_{22}^{13} = R_{112} \otimes_{22} R_{22}$. Again we use certain intermediate bimodules: $\widehat{\Lambda}_1 = R_{1111} \otimes_{112} R_{112} \otimes_{22} R_{22}$, $\widehat{\Lambda}_2 = R_{1111} \otimes_{211} R_{211} \otimes_{22} R_{22}$, $\widehat{\Lambda}_3 = R_{1111} \otimes_{211} R_{211} \otimes_{31} R_{211} \otimes_{22} R_{22}$ and $\widehat{\Lambda}_4 = R_{1111} \otimes_{121} R_{121} \otimes_{31} R_{211} \otimes_{22} R_{22}$ (see Figure 8).

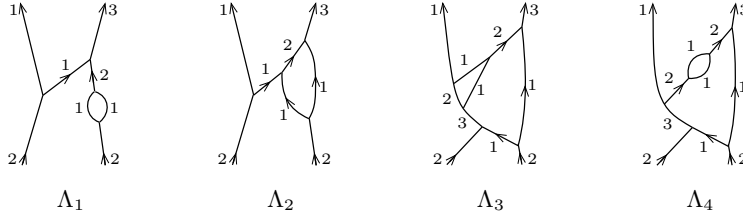


FIGURE 8. Intermediate webs $\Lambda_1, \dots, \Lambda_4$

We define ϕ_2 as the composite

$$\widehat{\mathcal{R}}_{22}^{13} \xrightarrow{\phi_2^1} \widehat{\Lambda}_1 \xrightarrow{\phi_2^2} \widehat{\Lambda}_2 \xrightarrow{\phi_2^3} \widehat{\Lambda}_3 \xrightarrow{\phi_2^4} \widehat{\Lambda}_4 \xrightarrow{\phi_2^5} \widehat{\mathcal{R}}_{2113}$$

with

$$\begin{aligned} \phi_1^1(a \otimes b) &= 1 \otimes a \otimes b, \\ \phi_1^2(a \otimes b \otimes c) &= ab \otimes 1 \otimes c, \\ \phi_1^3(a \otimes b \otimes c) &= a \otimes \Delta_{21}^{x_1 x_2 x_3}(b) \otimes c, \\ \phi_1^4(a \otimes b \otimes c \otimes d) &= ab \otimes 1 \otimes c \otimes d \end{aligned}$$

and with ϕ_1^5 defined by the map $R_{1111} \rightarrow R_{121}\{2\}$, which is the projection onto the second direct summand in the decomposition $R_{1111} \cong R_{121} \oplus x_2 R_{121}$.

Conversely, we define ψ_2 as the composite

$$\widehat{\mathcal{R}}_{2113} \xrightarrow{\psi_2^1} \widehat{\Lambda}_4 \xrightarrow{\psi_2^2} \widehat{\Lambda}_3 \xrightarrow{\psi_2^3} \widehat{\Lambda}_2 \xrightarrow{\psi_2^4} \widehat{\Lambda}_1 \xrightarrow{\psi_2^5} \widehat{\mathcal{R}}_{22}^{13}$$

with

$$\begin{aligned} \psi_1^1(a \otimes b \otimes c) &= 1 \otimes a \otimes b \otimes c, \\ \psi_1^2(a \otimes b \otimes c \otimes d) &= ab \otimes 1 \otimes c \otimes d, \\ \psi_1^3(a \otimes b \otimes c \otimes d) &= a \otimes bc \otimes d, \\ \psi_1^4(a \otimes b \otimes c) &= ab \otimes 1 \otimes c \end{aligned}$$

and with ψ_1^5 defined by the map $R_{1111} \rightarrow R_{112}\{2\}$, which is the projection onto the second direct summand in the decomposition $R_{1111} \cong R_{112} \oplus x_3 R_{121}$. A simple calculation shows that $\psi_2 \phi_2 = \text{id}$.

One can also easily check that $\psi_2 \phi_1 = 0$ and $\psi_1 \phi_2 = 0$. This shows that (ϕ_1, ϕ_2) is injective. To show that (ϕ_1, ϕ_2) is surjective note that both maps are left R_{13} -module maps and that the source and the target are both free R_{13} -modules of rank 9 with the same gradings. \square

4. THE LINK HOMOLOGY

Let us define the coloured HOMFLY-PT homology for links with components labelled 1 and 2. We use a similar setup to the one in [6]. To each braid diagram we associate a complex of bimodules (defined below) obtained from the categorified MOY calculus. This complex is invariant up to homotopy under the braidlike Reidemeister II and III moves. Then we take the Hochschild homology of each bimodule in the complex, which corresponds to the categorification of the Markov trace. This induces a complex of vector spaces whose homology is the one we are looking for. The latter is still invariant under the second and third Reidemeister

moves, because the Hochschild homology is a covariant functor, and also under the Markov moves, as we will show. Therefore we obtain a triply graded link homology. By taking the graded dimensions of the homology groups we get a triply graded link polynomial.

To define the complex of bimodules associated to a braid, it suffices to define it for a positive and for a negative crossing only. For an arbitrary braid one then tensors these complexes over all crossings. To each crossing with both strands labelled by 2, we associate a complex with three terms. For a positive, resp. negative, crossing between strands labelled 2, the terms in the complex are $\widehat{\uparrow}_{22}\{6\} \rightarrow \widehat{\mathfrak{H}}_{3111}\{2\} \rightarrow \widehat{\chi}_{22}$, resp. $\widehat{\chi}_{22}\{-8\} \rightarrow \widehat{\mathfrak{H}}_{3111}\{-8\} \rightarrow \widehat{\uparrow}_{22}\{-6\}$ (see Figures 9-10).

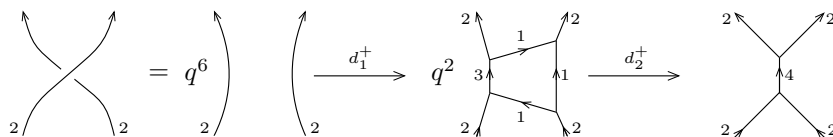


FIGURE 9. The complex of a positive crossing

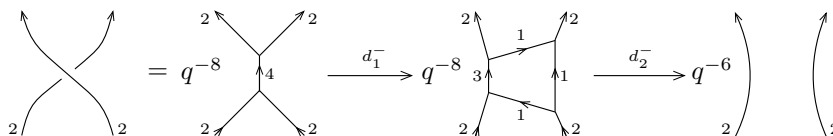


FIGURE 10. The complex of a negative crossing

In both cases, the cohomological degree is fixed by putting the bimodule $\widehat{\chi}_{22}$ in cohomological degree 0.

To define the differentials we need the intermediate webs Φ, Ψ and Ω (see Figure 11). Note that $\widehat{\Psi} \cong \widehat{\Omega}$.

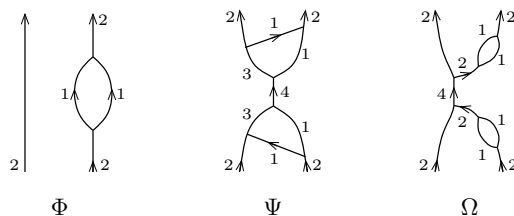


FIGURE 11. Intermediate webs Φ, Ψ and Ω

The differential d_1^+ is the composite of

$$\widehat{\uparrow}_{22} \xrightarrow{d_{11}^+} \widehat{\Phi} \xrightarrow{d_{12}^+} \widehat{\mathfrak{H}}_{3111}\{-4\},$$

where d_{11}^+ is defined by the inclusion $R_{22} \hookrightarrow R_{211}$ to create the digon and

$$d_{12}^+(a \otimes b) = \Delta_{21}^{x_1, x_2, x_3}(a) \otimes b.$$

The differential d_2^+ is the composite of

$$\widehat{\mathfrak{H}}_{3111}\{-4\} \xrightarrow{d_{21}^+} \widehat{\Psi}\{-10\} \cong \widehat{\Omega}\{-10\} \xrightarrow{d_{22}^+} \widehat{\chi}_{22}\{-6\},$$

where d_{21}^+ is defined by

$$d_{21}^+(a \otimes b \otimes c) = a \cdot \Delta_{31}(b) \otimes c$$

and d_{22}^+ by twice applying the map $R_{211} \rightarrow R_{22}\{2\}$ given by the projection onto the second direct summand in the decomposition $R_{211} = R_{22} \oplus x_3 R_{22}$. Direct computation shows that $d_2^+ d_1^+ = 0$.

The first differential in the complex associated to a negative crossing is the composite of

$$\widehat{\chi}_{22} \xrightarrow{d_{11}^-} \widehat{\Omega} \cong \widehat{\Psi} \xrightarrow{d_{12}^-} \widehat{\mathfrak{H}}_{3111},$$

where d_{11}^- is defined by twice applying the inclusion $R_{22} \hookrightarrow R_{211}$ to create the digons and

$$d_{12}^-(a \otimes b \otimes c \otimes d) = a \otimes b c \otimes d.$$

The differential d_2^- is the composite of

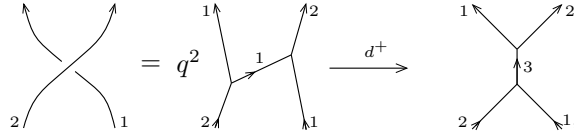
$$\widehat{\mathfrak{H}}_{3111} \xrightarrow{d_{21}^-} \widehat{\Phi} \xrightarrow{d_{22}^-} \widehat{\uparrow}_{22}\{2\},$$

where d_{21}^- is defined by

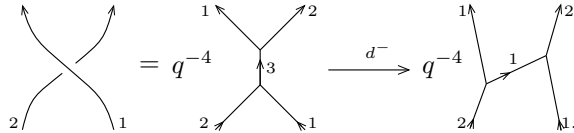
$$d_{21}^-(a \otimes b \otimes c) = a b \otimes c$$

and d_{22}^- by applying the same projection $R_{211} \rightarrow R_{22}\{2\}$ as above. Again, direct computation shows that $d_2^- d_1^- = 0$. Note that this calculation is much easier than the one which shows $d_2^+ d_1^+ = 0$. The latter is a direct consequence of the former by the observation that all maps involved in the definition of $d_2^\pm d_1^\pm$ are units and counits of the two biadjoint functors given by induction and restriction. As a matter of fact, whenever we use a unit in the definition of one, we use the corresponding counit in the definition of the other. Therefore the fact that $d_2^- d_1^- = 0$ implies $d_2^+ d_1^+ = 0$.¹

Next we define a complex of bimodules associated to a crossing of a strand labelled 1 and a strand labelled 2. To a positive crossing we associate the complex



and to a negative one



Again, in both cases, we put the bimodule $\widehat{\chi}_{21}^{12}$ in the cohomological degree 0. Note that $\widehat{\mathfrak{H}}_{21}^{12} = R_{111} \otimes_{21} R_{21}$ and $\widehat{\chi}_{21}^{12} = R_{12} \otimes_3 R_{21}$. We use the intermediate bimodules $\widehat{\Lambda}_1 = R_{111} \otimes_{21} R_{21} \otimes_3 R_{21} \cong R_{111} \otimes_{12} R_{12} \otimes_3 R_{21} = \widehat{\Lambda}_2$ (see Figure 12).

Then d^+ is the composite of

$$\widehat{\mathfrak{H}}_{21}^{12} \xrightarrow{d_1^+} \widehat{\Lambda}_1\{-4\} \xrightarrow{\cong} \widehat{\Lambda}_2\{-4\} \xrightarrow{d_2^+} \widehat{\chi}_{21}^{12}\{-2\}$$

¹We thank Mikhail Khovanov for this observation.

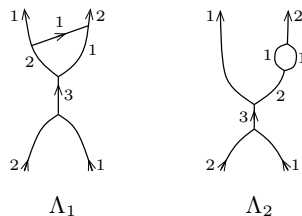


FIGURE 12. Intermediate webs Λ_1 and Λ_2

with

$$d_1^+(a \otimes b) = a \otimes \Delta_{21}(b),$$

$$a \otimes b \otimes c \xrightarrow{\cong} ab \otimes 1 \otimes c$$

and with d_2^+ defined by the map $R_{111} \rightarrow R_{12}\{2\}$ corresponding to the projection onto the second direct summand in the decomposition $R_{111} \cong R_{12} \oplus x_2 R_{12}$. It is easy to compute the image of d^+ on generators of the $R_{21} - R_{12}$ bimodule $R_{21} \otimes R_{111}$:

$$d^+(1 \otimes 1) = x_1 \otimes 1 - 1 \otimes x_3 \quad \text{and} \quad d^+(x_2 \otimes 1) = 1 \otimes x_1 x_2 - x_2 x_3 \otimes 1.$$

Similarly we define d^- as the composite

$$\widehat{\mathcal{X}}_{21}^{12} \xrightarrow{d_1^-} \widehat{\Lambda}_2 \xrightarrow{\cong} \widehat{\Lambda}_1 \xrightarrow{d_2^-} \widehat{\mathcal{X}}_{21}^{12}$$

with

$$d_1^-(a \otimes b) = 1 \otimes a \otimes b,$$

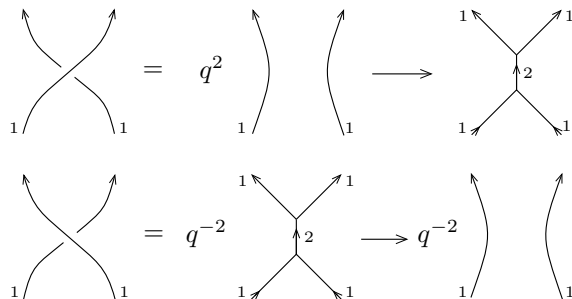
$$a \otimes b \otimes c \xrightarrow{\cong} ab \otimes 1 \otimes c,$$

$$d_2^-(a \otimes b \otimes c) = a \otimes bc.$$

Note that this yields $d^-(a \otimes b) = a \otimes b$.

We get similar complexes for the crossings with 1 and 2 swapped. The pictures can be obtained from the ones above by rotation around the y -axis, and the shifts are the same.

For a crossing with both strands labelled 1 we use the same complex of bimodules as Khovanov in [6]:



with \mathcal{X}_{11} in cohomological degree 0.

Let $HHH(D)$ denote the triply graded homology we defined above. Then to obtain the 1,2-coloured HOMFLY-PT homology $H_{1,2}(D)$ we have to apply some overall shifts. We define

Definition 4.1.

$$H_{1,2}(D) = HHH(D) \left\langle \frac{n_+^1 - n_-^1 - s_1(D) + 2n_+^2 - 2n_-^2 - 2s_2(D)}{2} \right\rangle \left\{ \frac{-n_+^1 + n_-^1 + s_1(D) - 2n_+^2 + 2n_-^2 + 2s_2(D)}{2}, \frac{-n_+^1 + n_-^1 + s_1(D) - 2n_+^2 + 2n_-^2 + 2s_2(D)}{2} \right\}.$$

The definitions of n_+^i, n_-^i and $s_i(D)$ were given in Section 2, $\langle j \rangle$ is an upward shift by j in the homological degree and $\{k, l\}$ denotes an upward shift by k in the Hochschild degree and by l in the q -degree.

Finally in the next two sections we prove the following.

Theorem 4.2. *For a given link L , $H_{1,2}(D)$ is independent of the chosen braid diagram D which represents it. Hence, $H_{1,2}(L)$ is a link invariant.*

5. INVARIANCE UNDER THE R2 AND R3 MOVES

The next thing to do is to prove invariance of the 1,2-coloured HOMFLY-PT homology under the second and third Reidemeister moves. If the strands involved are all labelled 1, we already have invariance by Khovanov’s [6] and Khovanov and Rozansky’s [8] results. If there are strands involved which are labelled 2, we will use a trick, inspired by the analogous trick in [9], which reduces to the case with all link components labelled 1. The argument is slightly tricky, so let us explain the general idea first.

Suppose we have a braid B with n strands, all labelled 2. Create a digon $\hat{\phi}_{11}$ on top of each strand. Our link homology complex corresponding to this braid with digons is “ $(1 + q^2)^n$ times” our link homology complex corresponding to B , which means that the latter complex is the direct sum of 2^n copies of the former with grading shifts according to the powers of q in the polynomial $(1 + q^2)^n$. We will prove in Lemma 5.2 that sliding the lower parts of the digons past the crossings is a grading preserving homotopy equivalence. After sliding this way the lower parts of all digons past all crossings (see Figure 13), we obtain a braided diagram B' which is the 2-cable of B with the two top endpoints, and the two bottom endpoints respectively, of each cable zipped together. The complex of bimodules

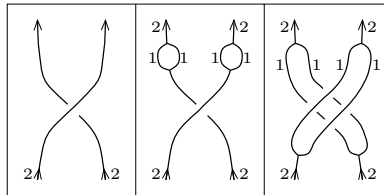


FIGURE 13. Creating and sliding digons

associated to B' is the tensor product of the HOMFLY-PT complex associated to the 2-cable of B with two complexes, one associated to the top endpoints of the cables and one to the bottom endpoints. Performing a Reidemeister II or III move on B corresponds to performing a series of Reidemeister II and III moves on its

2-cable. By Khovanov and Rozansky's [8] results we know that the complex of bimodules associated to the 2-cable is invariant up to homotopy equivalence under Reidemeister II and III moves. Therefore the complex of bimodules associated to B' is invariant under the Reidemeister II and III moves up to homotopy equivalence. In the next paragraph we will show that the category \mathfrak{C}' whose objects are complexes of graded bimodules, such that its homogeneous summands are finite-dimensional for each degree and whose morphisms are graded maps of complexes modulo homotopy, is Krull-Schmidt, i.e. all objects have a unique decomposition into indecomposables. Therefore, the fact that B' is invariant under the braidlike Reidemeister II and III moves implies that B is invariant under the same moves, because the former is “ $(1 + q^2)^n$ times” the latter.

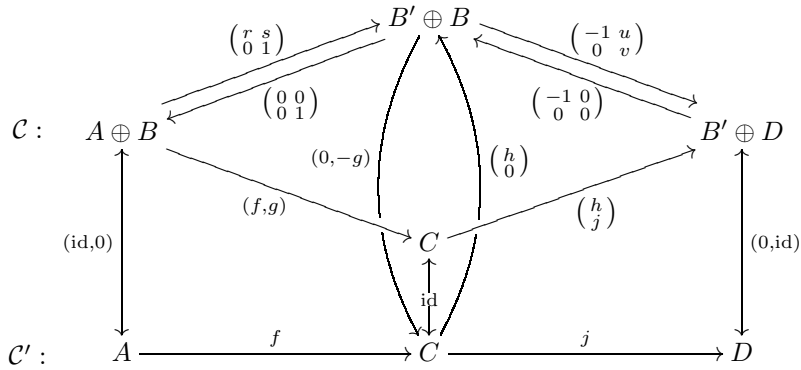
Let us now prove that \mathfrak{C}' is Krull-Schmidt. First of all, this category is additive with finite-dimensional hom-spaces, so it suffices to show that it is Karoubian [2], i.e. each idempotent splits. Note that the category \mathfrak{C} whose objects are complexes of graded bimodules, such that its homogeneous summands are finite-dimensional for each degree and whose morphisms are graded maps of complexes, is an Abelian category with finite-dimensional hom-spaces. Therefore it is Krull-Schmidt (Theorem 2.18 in [12]). Note also that any complex decomposes uniquely into a maximal contractible subcomplex and its complement. Now, let M be an object in \mathfrak{C}' and $e \in A' = \text{End}_{\mathfrak{C}'}(M)$ be an idempotent. By the above, we can assume that M has no contractible summands in \mathfrak{C} . Note that $A = \text{End}_{\mathfrak{C}}(M)$ is Artinian, because it is finite-dimensional, so its Jacobson radical J is nilpotent by Theorem 1.2.7 of [1]. Note also that the ideal $N \subset A$ of null-homotopic maps is contained in J . If M is indecomposable, this is true because A is local by Lemma 1.4.5 in [1]. If M is a finite direct sum of indecomposables, then A is the direct sum of Hom-spaces between the indecomposable summands. Write J in terms of the Jacobson radicals of the endomorphism rings of the indecomposables using Proposition 1.2.5 of [1]. Again we see that $N \subset J$. Therefore, the nilpotency of J implies that of N . By Theorem 1.7.3 in [1] this implies that we can lift e from $A' = A/N$ to A . Since \mathfrak{C} is Krull-Schmidt, we can split e in \mathfrak{C} . Of course that splitting descends to a splitting in \mathfrak{C}' , which is what we had to prove.²

Alternatively, one could note that the Poincaré polynomial of the 1,2-coloured HOMFLY-PT homology of B' is $(1 + q^2)^n$ times that of B and that therefore the invariance of the former under the braidlike Reidemeister II and III implies the invariance of the latter. Note that this alternative argument is really different from the one in the previous two paragraphs. The Poincaré polynomials of the 1,2-coloured HOMFLY-PT homology of B and B' can only be obtained after taking the Hochschild homology of B and B' , whereas the argument of the previous two paragraphs applies to B and B' directly, before taking their Hochschild homology. Either way, note that this digon trick does not give a specific homotopy equivalence between the complexes before and after a Reidemeister move. It only shows that the corresponding Poincaré polynomials are equal. This would be a problem if we wanted to show functoriality under link cobordisms of the whole construction. However, even the ordinary HOMFLY-PT-homology by Khovanov and Rozansky has not been proven to be functorial and probably is not.

²We thank Mikhail Khovanov for explaining the results in this paragraph to us.

Before we prove the crucial lemma we have to prove an auxiliary result. In the following lemma the top and the bottom of the diagram are complexes. The reader can easily check the following auxiliary result.

Lemma 5.1. *The diagram below gives a homotopy equivalence between the top and the bottom complex.*

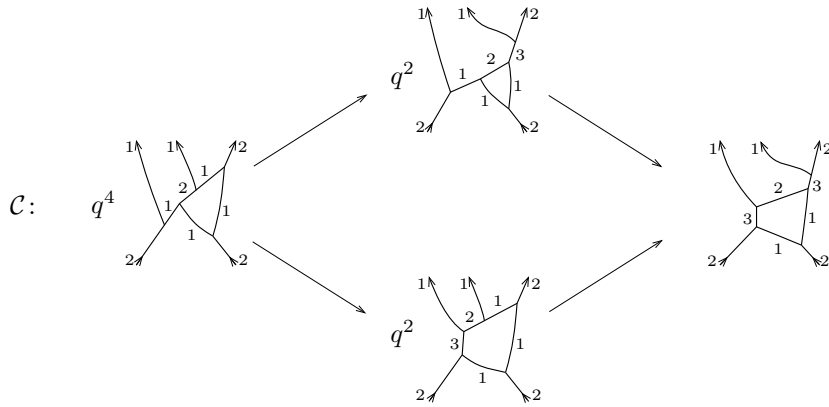


Using the lemma above we can now prove the following crucial lemma.

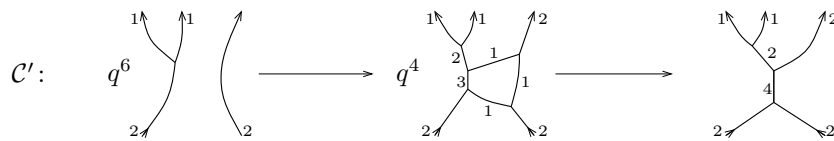
Lemma 5.2. *We have the homotopy equivalence*

(5.1)

Proof. Note that the complex of bimodules \mathcal{C} associated to the r.h.s. of Equation (5.1) is given by



The complex \mathcal{C}' associated to the l.h.s. is given by



By Lemma 3.4 we have

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 1 \end{array} & \begin{array}{c} \uparrow 2 \\ \swarrow 1 \\ \downarrow 1 \end{array} \\
 \cong & q^2 & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 2 \end{array} \\
 \end{array} \cong \begin{array}{c} \uparrow 1 \\ \downarrow 2 \end{array} \oplus \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 A & & B
 \end{array}$$

By Lemma 3.1 we have

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 1 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \cong & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 1 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \oplus q^2 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 \end{array} \cong \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \oplus q^2 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 B' & & B
 \end{array}$$

and by Lemma 3.6 we have

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \cong & \begin{array}{c} \uparrow 1 \\ \swarrow 3 \\ \downarrow 4 \end{array} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \oplus q^2 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 \end{array} \cong \begin{array}{c} \uparrow 1 \\ \swarrow 3 \\ \downarrow 4 \end{array} \oplus q^2 \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 B' & & D
 \end{array}$$

Note that

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 4 \end{array} & \cong & \begin{array}{c} \uparrow 1 \\ \swarrow 3 \\ \downarrow 4 \end{array} \\
 \end{array}
 \end{array}$$

Finally apply Lemma 5.1, which is justified because

- (1) $d: A \rightarrow B$ is zero,
- (2) $d: B' \rightarrow D$ is zero,
- (3) $d: B \rightarrow B$ is the identity,
- (4) $d: B' \rightarrow B'$ is minus the identity,

$$\begin{array}{c}
 (5) \quad \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \downarrow 2 \end{array} & \longrightarrow & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \cong & & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \end{array} \xrightarrow{d} \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \oplus q^{-2} \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \longrightarrow q^{-2} \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \downarrow 2 \end{array} & \xrightarrow{d} & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \cong & & \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 2 \end{array} \\
 \end{array} \xrightarrow{d} q^{-2} \begin{array}{c} \uparrow 1 \\ \swarrow 1 \\ \downarrow 3 \end{array}
 \end{array}$$

and

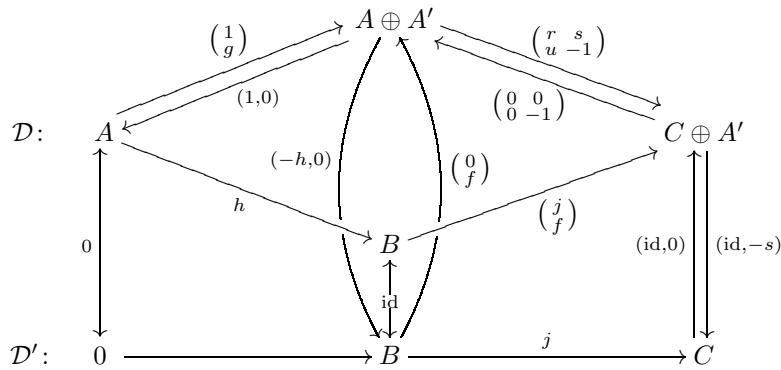
$$\begin{array}{c}
 (6) \quad \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} & \longrightarrow & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \\
 \cong & & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \\
 \end{array} \xrightarrow{d} \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \oplus q^{-4} \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \longrightarrow q^{-4} \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} & \xrightarrow{d} & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \\
 \cong & & \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array} \\
 \end{array} \xrightarrow{d} q^{-4} \begin{array}{c} \uparrow 1 \\ \swarrow 2 \\ \downarrow 3 \end{array}
 \end{array}$$

All assertions follow from straightforward computations and can easily be checked. For the first assertion one only has to compute the image of $1 \otimes 1$, which is indeed zero. For the second, third and fourth it suffices to compute the images of $1 \otimes 1 \otimes 1$ and $1 \otimes x_1 \otimes 1$. For the fifth it suffices to compare the images of $1 \otimes 1$. For the sixth, one has to compare the images of $1 \otimes 1 \otimes 1 \otimes 1$, $1 \otimes 1 \otimes x_3 \otimes 1$ and $x_3 \otimes 1 \otimes 1 \otimes 1$. \square

Of course there are also homotopy equivalences analogous to the one in Lemma 5.2 for a negative crossing or an 11-splitting of the right top strand.

In the following lemma the top and bottom parts are complexes again.

Lemma 5.3. *If $sf = 0$, the diagram below defines a homotopy equivalence between the top and the bottom complex.*



Proof. All maps are indicated in the figure, so the reader can check everything easily. Two identities are helpful: since the top of the figure is a complex, we have

$$-jh = r + sg \quad \text{and} \quad fh = g - u. \quad \square$$

Lemma 5.4. *We have the homotopy equivalence*

$$(5.2) \quad \begin{array}{c} \text{1} \nearrow \text{1} \\ \text{1} \searrow \text{2} \end{array} \cong \begin{array}{c} \text{1} \nearrow \text{1} \\ \text{1} \searrow \text{2} \end{array}.$$

Proof. The complex associated to the l.h.s., denoted \mathcal{D}' , is given in Figure 14. The

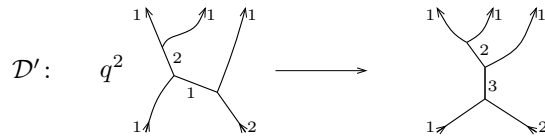


FIGURE 14. Complex of the l.h.s. of Equation (5.2)

complex associated to the r.h.s., denoted \mathcal{D} , is given in Figure 15.

This time we have

$$\begin{array}{c} \text{1} \nearrow \text{1} \\ \text{1} \searrow \text{2} \end{array} \cong \begin{array}{c} \text{1} \nearrow \text{1} \\ \text{1} \searrow \text{2} \end{array} \oplus q^2 \begin{array}{c} \text{1} \nearrow \text{1} \\ \text{1} \searrow \text{2} \end{array}$$

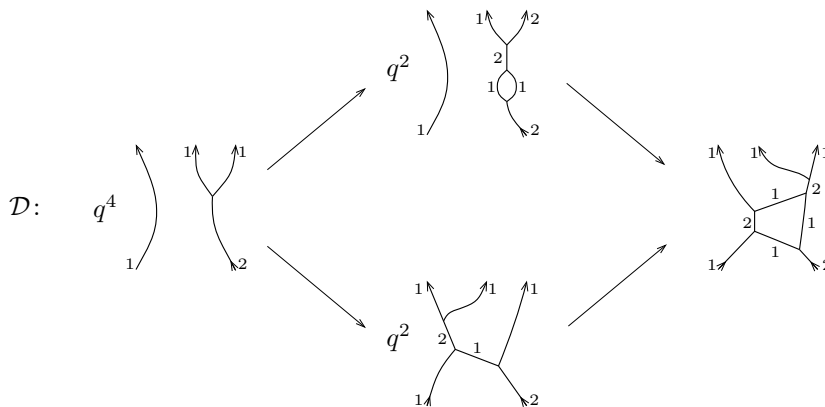
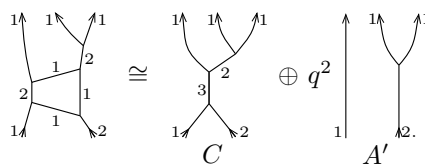
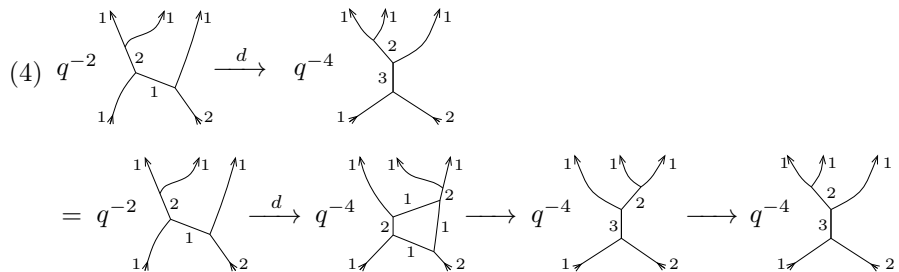


FIGURE 15. Complex of the r.h.s. of Equation (5.2)

and



- (1) f is equal to the identity,
- (2) u is equal to minus the identity,
- (3) $sf = 0$ and



Since we have given all the maps, the reader can check the claims by straightforward computations. Note that for the first two claims it suffices to compute the image of $1 \otimes 1$. For the last two claims one has to compute the images of $1 \otimes 1 \otimes 1$ and $1 \otimes x_2 \otimes 1$. \square

Again, there are analogous homotopy equivalences for a negative crossing or if one swaps the 1- and 2-strands in the lemma above.

6. INVARIANCE UNDER THE MARKOV MOVES

6.1. Hochschild homology of bimodules as the homology of a Koszul complex of polynomial rings. The Hochschild homology of a bimodule over the polynomial ring can be obtained as the homology of a corresponding Koszul complex of certain polynomial rings in many variables. This idea was explained and used by Khovanov in [6]. Here we shall briefly describe how to extend it to our case.

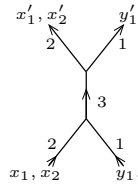
First of all, we change our polynomial notation in this section. Namely, the polynomial ring $R_{i_1 \dots i_k}$ can be represented as the polynomial ring in the variables which are the i_1 elementary symmetric polynomials in the first i_1 variables, the i_2 elementary symmetric polynomials in the following i_2 variables, etc., and the i_k elementary symmetric polynomials in the last i_k variables. In this section we will always work with these “new” variables, i.e. the elementary symmetric polynomials, because it is more convenient for our purposes here. Thus to each strand labelled by k we associate the k variables x_1, \dots, x_k , such that the degree of x_i is equal to $2i$ for $i = 1, \dots, k$.

To begin with, we describe which polynomial ring to associate to a web. Take a web and choose a height function to separate the vertices according to height. This way chop up the web into several layers with only one vertex. To each layer we associate a new set of variables. To each vertex with incident edges labelled i, j and $i + j$, we associate the $i + j$ polynomials which are the differences of the k -th symmetric polynomials in the outgoing variables and the incoming variables for every $k = 1, \dots, i + j$. Moreover, if there are two different sets of variables x_1, \dots, x_k and x'_1, \dots, x'_k associated to a given edge labelled k , we also associate the k polynomials $x_i - x'_i$ for all $i = 1, \dots, k$. Finally, to the whole graph we associate the polynomial ring in all the variables modded out by the ideal generated by all the polynomials associated to the vertices and edges.

There is an isomorphism between these polynomial rings and the corresponding bimodules associated to the graph. Indeed, to the tensor product $p(\underline{x}) \otimes q(\underline{x})$ in variables \underline{x} there corresponds the polynomial $p(\underline{x}')q(\underline{x})$, where the variables \underline{x} are of the bottom layer and \underline{x}' are of the top layer. Loosely speaking, the position of the factor in the tensor product corresponds to the same polynomial in the variables corresponding to the layer.

Let us do an example:

Example 6.1. Consider the web χ_{21} :



The polynomial ring associated to this web is the ring

$$P_{\chi_{21}} := \mathbb{C}[x_1, x_2, y_1, x'_1, x'_2, y'_1].$$

The ideal $I_{\chi_{21}}$ by which we have to quotient is generated by the differences of the symmetric polynomials in all top and bottom variables (since the middle edge is labelled by 3). There are three elementary symmetric polynomial in this case: $\Sigma_1 = x_1 + y_1$, $\Sigma_2 = x_2 + x_1y_1$ and $\Sigma_3 = x_2y_1$, and so

$$I_{\chi_{21}} = \langle x_1 + y_1 - x'_1 - y'_1, x_2 + x_1y_1 - x'_2 - x'_1y'_1, x_2y_1 - x'_2y'_1 \rangle.$$

Hence, the polynomial ring we associate to it is given by

$$R_{\chi_{21}} = P_{\chi_{21}}/I_{\chi_{21}}.$$

On the other hand, the bimodule $\widehat{\mathcal{X}}_{21}$ associated to the web \mathcal{X}_{21} is $R_{21} \otimes_3 R_{21}$, and its elements are a linear combination of elements of the form $p(x_1, x_2, y_1) \otimes q(x_1, x_2, y_1)$ for some polynomials p and q . Finally, the isomorphism between $\widehat{\mathcal{X}}_{21}$ and $R_{\mathcal{X}_{21}}$ is given by

$$p(x_1, x_2, y_1) \otimes q(x_1, x_2, y_1) \longleftrightarrow p(x'_1, x'_2, y'_1)q(x_1, x_2, y_1).$$

In such a way we have obtained the bijective correspondence between the bimodules $\widehat{\Gamma}$ and the polynomial rings R_Γ that are associated to the open trivalent graph Γ . The closure of Γ in the bimodule picture corresponds to taking the Hochschild homology of $\widehat{\Gamma}$. In the polynomial ring picture this is isomorphic to the homology of the Koszul complex over R_Γ which is the tensor product of the complexes

$$(6.1) \quad 0 \longrightarrow R_\Gamma\{-1, 2i - 1\} \xrightarrow{x_i - x'_i} R_\Gamma \longrightarrow 0,$$

where the x_i 's are the bottom layer variables and the x'_i 's are the top layer variables. We put the right-hand side term in (co)homological degree zero. The first shift is in the Hochschild (homological) degree, and the second one is in the q -degree, so that the maps have bi-degree $(1, 1)$.

6.2. Invariance under the first Markov move. Essentially we have to show that the Hochschild homology of the tensor product $B_1 \otimes B_2$ of the bimodules B_1 and B_2 is isomorphic to the Hochschild homology of $B_2 \otimes B_1$, i.e.

$$(6.2) \quad HH(B_1 \otimes B_2) = HH(B_2 \otimes B_1).$$

We shall prove this by passing to the polynomial ring and Koszul complex description from above. Recall that to the open trivalent graph Γ we have associated the polynomial algebra P_Γ (the ring of polynomials in all variables) quotiented by the ideal I_Γ generated by certain polynomials. Since for each layer we introduced new variables, the sequence of these polynomials is regular, and so we have that $R_\Gamma = P_\Gamma/I_\Gamma$ is the homology of the Koszul complex obtained by tensoring together all complexes of the form

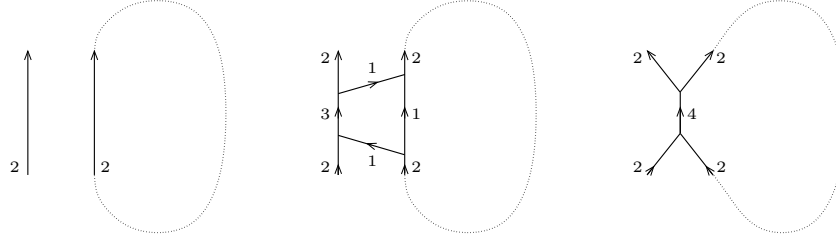
$$0 \longrightarrow P_\Gamma \xrightarrow{f} P_\Gamma \longrightarrow 0$$

for $f \in I_\Gamma$. We call this the Koszul complex generated by I_Γ .

Finally, the object associated to the closure of the graph Γ is given by the homology of the Koszul complex defined by (6.1), so it is isomorphic to the homology of the Koszul complex generated by the polynomials which define I_Γ together with the polynomials $x_i - x'_i$ which come from the closure. If Γ is the vertical glueing of Γ_1 and Γ_2 , then in I_Γ we have the polynomials $y_j - y'_j$ which correspond to the edges which are glued together. The Hochschild homology of $\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2$ is isomorphic to the homology generated by $I_{\Gamma_1}, I_{\Gamma_2}$ and the polynomials $x_i - x'_i$ and $y_j - y'_j$. Clearly the same holds for $\widehat{\Gamma}_2 \otimes \widehat{\Gamma}_1$, with the role of the $x_i - x'_i$ and $y_j - y'_j$ interchanged. Thus we have proved (6.2).

6.3. Invariance under the second Markov move. Invariance under the second Markov move corresponds to invariance under the Reidemeister move I. If the strand involved is labelled 1, the result was proved by Khovanov and Khovanov and Rozansky [6, 8] (see below).

It remains to show invariance if the strand is labelled 2. For both the positive and the negative crossing, we have the same three resolutions:



For each of these, we shall give its description in a polynomial language and compute the homology of the corresponding Koszul complex.

The resolution $\uparrow\uparrow$: Before closing the right strand, we have an open graph. To the bottom layer we associate variables x_1 and x_2 to the left strand, and y_1 and y_2 to the right strand, while to the top layer we associate variables x'_1 and x'_2 to the left strand, and y'_1 and y'_2 to the right strand. Then the ring $R_{\uparrow\uparrow}$, which we associate to it, is the ring of polynomials in all these variables modded out by the ideal generated by the polynomials

$$x'_1 - x_1, \quad x'_2 - x_2, \quad y'_1 - y_1, \quad y'_2 - y_2,$$

i.e. it is isomorphic to the ring

$$B := \mathbb{C}[x_1, x_2, y_1, y_2].$$

The resolution \varkappa : The variables that we associate to the bottom and the top layers are the same as above. To the bottom middle strand we associate the variable z_1 , to the top middle strand we associate z'_1 and to the right strand we associate the variable t_1 . Then the corresponding ring R_{\varkappa} is the ring of polynomials in all these variables, modded out by the ideal generated by the polynomials

$$\begin{aligned} y_1 - z_1 - t_1, \quad y_2 - z_1 t_1, \quad y'_1 - z'_1 - t_1, \quad y'_2 - z'_1 t_1, \\ x_1 + z_1 - x'_1 - z'_1, \quad x_2 + x_1 z_1 - x'_2 - x'_1 z'_1, \quad x_2 z_1 - x'_2 z'_1. \end{aligned}$$

From the first four relations, we can exclude t_1 and obtain the quadratic relations for z_1 and z'_1 :

$$z_1^2 = y_1 z_1 - y_2 \quad \text{and} \quad z_1'^2 = y'_1 z'_1 - y'_2.$$

Hence, every element from R_{\varkappa} can be written as $a + bz_1 + cz'_1 + dz_1 z'_1$, where a, b, c and d are polynomials only in x 's and y 's (with or without primes).

The resolution \varkappa : The variables we associate to the bottom and top layers are again the same. This time we obtain the ring R_{\varkappa} by quotienting by the ideal generated by the following four polynomials:

$$\begin{aligned} x_1 + y_1 - x'_1 - y'_1, \quad x_2 + y_2 + x_1 y_1 - x'_2 - y'_2 - x'_1 y'_1, \\ x_2 y_1 + x_1 y_2 - x'_2 y'_1 - x'_1 y'_2, \quad x_2 y_2 - x'_2 y'_2. \end{aligned}$$

To the closure of the right strands of each of these graphs there corresponds the homology of the tensor product of the following two Koszul complexes:

$$\begin{aligned} 0 \longrightarrow R_{\Gamma}\{-1, 1\} \xrightarrow{y_1 - y'_1} R_{\Gamma} \longrightarrow 0, \\ 0 \longrightarrow R_{\Gamma}\{-1, 3\} \xrightarrow{y_2 - y'_2} R_{\Gamma} \longrightarrow 0. \end{aligned}$$

Hence, in all three cases we can have homology in three homological gradings, 0, -1 and -2. We denote this homology by $HH^R(\Gamma)$.

In the case of $\uparrow\uparrow$ we have that both differentials are 0. Hence

$$\begin{aligned} HH_0^R(\uparrow\uparrow) &= B = \mathbb{C}[x_1, x_2, y_1, y_2], \\ HH_{-1}^R(\uparrow\uparrow) &= B\{1\} \oplus B\{3\}, \\ HH_{-2}^R(\uparrow\uparrow) &= B\{4\}. \end{aligned}$$

For the other two cases the computations are a bit more involved, and we want to explain the general idea first. The Hochschild homology is the homology of a complex which is the tensor product of complexes of the form

$$0 \longrightarrow P/I \xrightarrow{p} P/I \longrightarrow 0,$$

where P is a polynomial ring, I is an ideal and $p \in R$ is a polynomial. Let us explain how to compute the homology of one such complex.

The main part is the computation of the kernel and the cokernel of the map above. The cokernel is easily computed and is equal to the quotient ring $P/\langle I, p \rangle$. Now, let us pass to the kernel. For any polynomial $q \in P$ to be in the kernel, i.e. to be a cocycle, we must have $pq \in I$. In other words, we have to compute the colon ideal $Q = (I : pP)$. In the cases we are interested in, Q is always a principal ideal, i.e. $Q = qP$ for some $q \in P$. Then the kernel is given by $Q/I = qP/I$. However, this is isomorphic to $Q/\langle I, p \rangle$, since $pq \in I$, which is the form in which we present the results below.

It is not hard to extend these calculations to a tensor product of complexes of the above form. In particular, all homologies are certain ideals, modded out by the ideal generated by I_Γ and the polynomials $y_i - y'_i$, $i = 1, 2$, which in particular includes the polynomials $x_i - x'_i$, $i = 1, 2$, and also $z_1 - z'_1$ in the case of the web \varkappa . Hence, in the homology, all variables with primes are equal to the corresponding variables without the primes.

In the case of \varkappa we have that $y_2 - y'_2 = t_1(z_1 - z'_1) = (y_1 - z_1)(y_1 - y'_1)$, and straightforward computations along the lines sketched above give

$$\begin{aligned} HH_0^R(\varkappa) &= A = \mathbb{C}[x_1, x_2, y_1, y_2, z_1] / \langle z_1^2 - y_1 z_1 + y_2 \rangle, \\ HH_{-1}^R(\varkappa) &= \{((y_1 - z_1)g + (x_2 - y_2 - z_1(x_1 - y_1))h, g) \mid g, h \in A\} \subset A\{1\} \oplus A\{3\}, \\ HH_{-2}^R(\varkappa) &= (x_2 - y_2 - z_1(x_1 - y_1))A\{4\}. \end{aligned}$$

Note that we have $A \cong B \oplus z_1 B$.

Finally, in the case of \varkappa , we obtain

$$(6.3) \quad HH_0^R(\varkappa) = B = \mathbb{C}[x_1, x_2, y_1, y_2],$$

$$(6.4) \quad HH_{-1}^R(\varkappa) = \{(-c(x_2 - y_2) + (cy_1 + dy_2)(x_1 - y_1), c(x_1 - y_1) + d(x_2 - y_2)) \mid c, d \in B\} \subset B\{1\} \oplus B\{3\},$$

$$(6.5) \quad HH_{-2}^R(\varkappa) = pB\{4\},$$

where $p = (x_2 - y_2)^2 + (x_1 - y_1)(x_1 y_2 - x_2 y_1)$.

Now, we pass to the differentials. In our polynomial notation, in the case of the positive crossing, the maps are (up to a non-zero scalar)

$$R_{\uparrow\uparrow} \rightarrow R_{\varkappa}\{-4\} : 1 \mapsto (x_2 + x'_2 - y_2 - y'_2) + (y_1 - x'_1)z_1 + (y'_1 - x_1)z'_1.$$

We recall that $\langle i \rangle$ denotes the upward shift by i in the homological degree, while $\{k, l\}$ denotes the upward shift by k in the Hochschild degree and by l in the q -degree.

For the Reidemeister I move involving a strand labelled 1, we get a similar shift (see [6, 8], or apply the same methods as above):

$$\begin{aligned}
 HHH \left(\begin{array}{c} \uparrow \\ \text{circle} \\ \downarrow \\ 1 \end{array} \right) &= HHH \left(\begin{array}{c} \uparrow \\ \phantom{\text{circle}} \\ \downarrow \\ 1 \end{array} \right), \\
 HHH \left(\begin{array}{c} \uparrow \\ \text{circle} \\ \downarrow \\ 1 \end{array} \right) &= HHH \left(\begin{array}{c} \uparrow \\ \phantom{\text{circle}} \\ \downarrow \\ 1 \end{array} \right) \langle 1 \rangle \{-1, -1\}.
 \end{aligned}$$

The overall shift in the definition of $H_{1,2}(D)$ compensates this behaviour under the Reidemeister I moves, and we get a genuine link invariant.

6.4. Categorification of the axioms (A1) and (A2). In this subsection we shall show that our construction categorifies the axioms (A1) and (A2) of the MOY calculus, which include the closures of the strands. The powers of q in (A1) and (A2) will correspond to the internal q -grading (grading of the polynomial ring), while the powers of t will correspond to the Hochschild grading.

First we focus on the (A1) axiom for arbitrary k . The circle is the closure of the single strand labelled with k . Denote the graph that consists of this strand by \uparrow_k and the variables that we associate to it by x_1, \dots, x_k (remember that in this section we assume that $\deg x_i = 2i$). Then to this \uparrow_k we associate the ring $R_{\uparrow_k} = \mathbb{C}[x_1, \dots, x_k]$, and to its closure the Hochschild homology $HH(\uparrow_k)$ which is the homology of the Koszul complex obtained by tensoring

$$0 \longrightarrow R_{\uparrow_k} \{-1, 2i - 1\} \xrightarrow{0} R_{\uparrow_k} \longrightarrow 0$$

for all $i = 1, \dots, k$. Hence we have

$$HH(\uparrow_k) = \bigotimes_{i=1}^k (\mathbb{C}[x_1, \dots, x_k] \oplus \mathbb{C}[x_1, \dots, x_k] \{-1, 2i - 1\}),$$

which categorifies axiom (A1).

Now we consider axiom (A2). The left-hand side of the axiom is the closure of the right strand of the graph, which we denoted by χ_{ij} . As we previously said, we are only interested in the cases when the indices i and j are from the set $\{1, 2\}$, and so we have four cases. Each of these cases we treat in the same way as the case of χ_{22} , which we dealt with in the previous subsection. We associate the variables x_1, \dots, x_i and x'_1, \dots, x'_i to the strands on the left-hand side and the variables y_1, \dots, y_j and y'_1, \dots, y'_j to the strands on the right-hand side, and compute the homology (that we denote by $HH^R(\chi_{ij})$) of the tensor product of the Koszul complexes

$$0 \longrightarrow R_{\chi_{ij}} \{-1, 2l - 1\} \xrightarrow{y_l - y'_l} R_{\chi_{ij}} \longrightarrow 0$$

for $l = 1, \dots, j$. In the remaining part of the graph, the variables y don't appear again, while as before, in the $HH^R(\chi_{ij})$ we have that $x_l = x'_l$, for all $l = 1, \dots, i$. Finally, the right-hand side of axiom (A2) is \uparrow_i , to which we associate the variables x_1, \dots, x_i , and consequently the ring $R_{\uparrow_i} = \mathbb{C}[x_1, \dots, x_i]$.

Now, in the four cases we are interested in, the homologies are the following:

$i = j = 1$: The corresponding ring in this case is $R_{\chi_{11}} = \mathbb{C}[x_1, x'_1, y_1, y'_1]/\langle x_1 + y_1 - x'_1 - y'_1, x_1 y_1 - x'_1 y'_1 \rangle$, while $HH_{\chi_{11}}^R$ is the homology of the complex

$$0 \longrightarrow R_{\chi_{11}}\{-1, 1\} \xrightarrow{y_1 - y'_1} R_{\chi_{11}} \longrightarrow 0.$$

Direct computation of this homology gives

$$\begin{aligned} HH_0^R(\chi_{11}) &= \mathbb{C}[x_1, y_1], \\ HH_{-1}^R(\chi_{11}) &= (x_1 - y_1)\mathbb{C}[x_1, y_1]\{1\}, \end{aligned}$$

which categorifies (A2) in this case.

$i = 2, j = 1$: The corresponding ring in this case is

$$R_{\chi_{21}} = \mathbb{C}[x_1, x_2, x'_1, x'_2, y_1, y'_1]/\langle x_1 + y_1 - x'_1 - y'_1, x_1 y_1 + x_2 - x'_1 y'_1 - x'_2, x_2 y_1 - x'_2 y'_1 \rangle,$$

while $HH_{\chi_{21}}^R$ is the homology of the complex

$$0 \longrightarrow R_{\chi_{21}}\{-1, 1\} \xrightarrow{y_1 - y'_1} R_{\chi_{21}} \longrightarrow 0.$$

Then we have

$$\begin{aligned} HH_0^R(\chi_{21}) &= \mathbb{C}[x_1, x_2, y_1], \\ HH_{-1}^R(\chi_{21}) &= (x_2 - x_1 y_1 + y_1^2)\mathbb{C}[x_1, x_2, y_1]\{1\}, \end{aligned}$$

as wanted.

$i = 1, j = 2$: The corresponding ring in this case is

$$R_{\chi_{12}} = \mathbb{C}[x_1, x'_1, y_1, y_2, y'_1, y'_2]/\langle x_1 + y_1 - x'_1 - y'_1, x_1 y_1 + y_2 - x'_1 y'_1 - y'_2, x_1 y_2 - x'_1 y'_2 \rangle,$$

while $HH_{\chi_{12}}^R$ is the homology of the complex obtained by tensoring

$$0 \longrightarrow R_{\chi_{12}}\{-1, 1\} \xrightarrow{y_1 - y'_1} R_{\chi_{12}} \longrightarrow 0$$

and

$$0 \longrightarrow R_{\chi_{12}}\{-1, 3\} \xrightarrow{y_2 - y'_2} R_{\chi_{12}} \longrightarrow 0.$$

Thus, we obtain

$$\begin{aligned} HH_0^R(\chi_{12}) &= \mathbb{C}[x_1, y_1, y_2], \\ HH_{-1}^R(\chi_{12}) &= \{(c(x_1 - y_1) + dy_2, -c + dx_1) \mid c, d \in \mathbb{C}[x_1, y_1, y_2]\} \\ &\quad \subset \mathbb{C}[x_1, y_1, y_2]\{1\} \oplus \mathbb{C}[x_1, y_1, y_2]\{3\}, \\ HH_{-2}^R(\chi_{12}) &= (x_1^2 - x_1 y_1 + y_2)\mathbb{C}[x_1, y_1, y_2]\{4\}, \end{aligned}$$

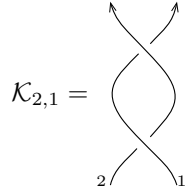
which categorifies (A2) in this case.

$i = j = 2$: We have already computed this case in the previous subsection in the formulas (6.3)-(6.5), which gives the categorification of (A2) in this case.

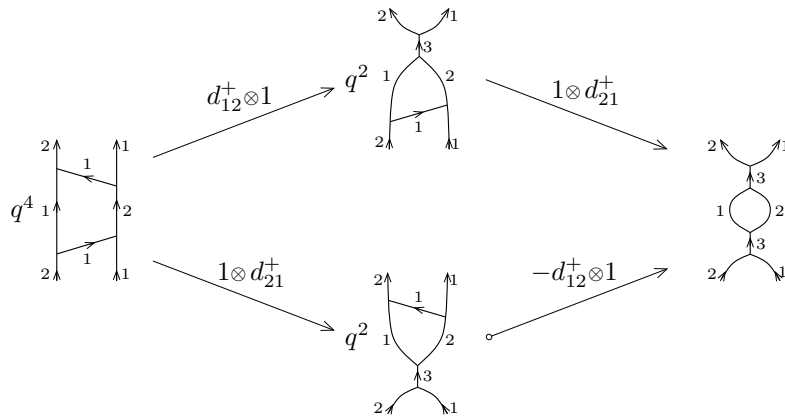
The categorification of axiom (A2) for general labellings is obtained in [11].

7. COMPUTATION OF THE 1,2-COLOURED HOPF LINK

Consider the following braid diagram labelled 1 and 2:



The complex of bimodules associated to $\mathcal{K}_{2,1}$ is



where d_{12}^+ and d_{21}^+ are as given in Section 4 and the rightmost term is in cohomological degree zero. The bimodules in cohomological degree -1 are both isomorphic to $\widehat{\chi}_{21} \oplus \widehat{\chi}_{21}\{2\}$ shifted up by 2. Using the direct sum decompositions (A3) and (A5) of Lemmas 3.1 and 3.4, one finds that the complex associated to \mathcal{K}_{21} is homotopy-equivalent to

$$(7.1) \quad \begin{array}{ccc} \begin{array}{c} \uparrow \\ q^6 \\ \uparrow \\ 2 \end{array} & \xrightarrow{\Delta_{21}} & \begin{array}{c} \begin{array}{c} \nearrow 1 \\ \searrow 2 \end{array} \\ q^2 \\ \begin{array}{c} \nwarrow 2 \\ \nearrow 1 \end{array} \end{array} \end{array} \xrightarrow{x_3 \otimes 1 - 1 \otimes x_3} \begin{array}{c} \begin{array}{c} \nearrow 1 \\ \searrow 2 \end{array} \\ \\ \begin{array}{c} \nwarrow 2 \\ \nearrow 1 \end{array} \end{array}$$

or, in the language of bimodules,

$$q^6 R_{21} \xrightarrow{\Delta_{21}} q^2 R_{21} \otimes_3 R_{21} \xrightarrow{x_3 \otimes 1 - 1 \otimes x_3} R_{21} \otimes_3 R_{21}.$$

Closing the strands of $\mathcal{K}_{2,1}$ gives the 1,2-coloured Hopf link $\mathcal{O}_{2,1}$. To obtain the HOMFLY-PT homology we have to take the Hochschild homology of the complex above. We use the notation and conventions of Section 6. More precisely, the bimodules R_{21} and $R_{21} \otimes_3 R_{21}$ become the polynomial rings $\mathbb{C}[x_1, x_2, y_1]$ and $R_{\chi_{21}}$, respectively (see Example 6.1). Then the complex becomes the following complex of polynomial rings:

$$q^6 \mathbb{C}[x_1, x_2, y_1] \xrightarrow{y_1^2 - x_1 y_1 + x_2} q^2 R_{\chi} \xrightarrow{y_1 - y_1'} R_{21} \otimes_3 R_{21}.$$

The Hochschild homology groups can be computed as at the end of Section 6.

The complex in Hochschild degree -3 is

$$q^{11}R_{21} \xrightarrow{1} q^{11}R_{21} \xrightarrow{0} q^9R_{21}$$

and has homology q^9R_{21} in cohomological degree 0 .

The complex in Hochschild degree -2 is

$$\begin{pmatrix} q^{10}R_{21} \\ q^8R_{21} \\ q^{10}R_{21} \end{pmatrix} \xrightarrow{\begin{pmatrix} \Delta_{21} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} q^6R_{21} \\ q^8R_{21} \\ q^{10}R_{21} \end{pmatrix} \xrightarrow{0} \begin{pmatrix} q^4R_{21} \\ q^6R_{21} \\ q^8R_{21} \end{pmatrix}$$

and has homology $q^6R_{21}/\Delta_{21}(1) \cong q^6R_{11}$ in cohomological degree -1 , and $q^4R_{21} \oplus q^6R_{21} \oplus q^8R_{21}$ in cohomological degree 0 .

The complex in Hochschild degree -1 is

$$\begin{pmatrix} q^7R_{21} \\ q^9R_{21} \\ q^7R_{21} \end{pmatrix} \xrightarrow{\begin{pmatrix} \Delta_{21} & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} q^3R_{21} \\ q^5R_{21} \\ q^7R_{21} \end{pmatrix} \xrightarrow{0} \begin{pmatrix} qR_{21} \\ q^3R_{21} \\ q^5R_{21} \end{pmatrix}$$

and has homology $q^3R_{11} \oplus q^5R_{11}$ in cohomological degree -1 , and $qR_{21} \oplus q^3R_{21} \oplus q^5R_{21}$ in cohomological degree 0 .

The complex in Hochschild degree 0 is

$$q^6R_{21} \xrightarrow{\Delta_{21}} q^2R_{21} \xrightarrow{0} R_{21}$$

and has homology q^2R_{11} in cohomological degree -1 , and R_{21} in cohomological degree 0 .

Therefore the Poincaré polynomial of $HHH(\mathbb{O}_{2,1})$ is

$$\begin{aligned} \mathcal{P}(\mathbb{O}_{2,1}) &= \frac{u^{-3/2}t^{3/2}q^{3/2}}{(1-q^2)^2} \left(u^{-1}(t^{-2}q^6 + t^{-1}(q^3 + q^5) + q^2) \right. \\ &\quad \left. + \frac{t^{-3}q^9 + t^{-2}(q^4 + q^6 + q^8) + t^{-1}(q + q^3 + q^5) + 1}{1-q^4} \right), \end{aligned}$$

where the variables t and u correspond to the Hochschild and the cohomological degrees, respectively, and we have included the normalization defined at the end of Section 4.

In [4] Gukov, Iqbal, Kozçaz and Vafa (GIKV) conjectured the existence of a triply graded link homology for links with components coloured by arbitrary representations of $sl(N)$. This conjecture is motivated by the physics of topological strings. They established a map between the refined topological vertex and the $sl(N)$ homological invariants of the Hopf link. When the components of the Hopf link are coloured with the first and second fundamental representations of $sl(N)$, the GIKV's superpolynomial is equal to

$$\begin{aligned} \bar{\mathcal{P}}_{(\square \boxplus)}(\mathbb{O})(\bar{u}, q, \mathbf{a}) &= \frac{1}{(1-q^2)^2} \left(\mathbf{a}^{-3} - \mathbf{a}^{-1}(q^{-2} + 1) + \mathbf{a}q^{-2} \right. \\ &\quad \left. + \bar{u}^2 \frac{\mathbf{a}^{-3}q^6 - \mathbf{a}^{-1}(q^2 + q^4 + q^6) + \mathbf{a}(1 + q^2 + q^4) - \mathbf{a}^3}{1-q^4} \right), \end{aligned}$$

where \bar{u} corresponds to the cohomological degree.

The change of variables

$$(\mathbf{a}^{-2}, \bar{u}^2) \rightarrow (-t^{-1}q, uq^{-3}t^{-1})$$

in $\bar{\mathcal{P}}_{(\square, \boxplus)}(\mathbb{O})(\bar{u}, q, \mathbf{a})$ gives a multiple of $\mathcal{P}(\mathbb{O}_{2,1})$:

$$\bar{\mathcal{P}}_{(\square, \boxplus)}(\mathbb{O})(u^{1/2}t^{-1/2}q^{-3/2}, q, it^{-1/2}q^{1/2}) = iu^{5/2}t^{-2}q^{-6}\mathcal{P}(\mathbb{O}_{2,1})(u, q, t),$$

the difference being due to different normalizations and conventions (cf. the values for the unknot).

8. CONJECTURES ABOUT THE HIGHER FUNDAMENTAL REPRESENTATIONS

Ideas similar to the ones in this paper will probably work for link components labelled with arbitrary MOY labels.

First of all, to the open MOY web Γ we associate the bimodule $\widehat{\Gamma}$ in the same way. In order to define the resolutions for the crossings, in principal one only has to know which maps to associate to the zip, the unzip, the digon creation and the digon annihilation. For the zip and the unzip our candidates are given in the following definition.

Definition 8.1. We define the linear maps

$$\mu_{ij}: \widehat{\chi}_{ij} \rightarrow \widehat{\uparrow}_{ij} \quad \text{and} \quad \Delta_{ij}: \widehat{\uparrow}_{ij} \rightarrow \widehat{\chi}_{ij}$$

by

$$\mu_{ij}(a \otimes b) = ab$$

and

$$\Delta_{ij}(1) = \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_i) \\ j \geq \alpha_1 \geq \dots \geq \alpha_i \geq 0}} (-1)^{|\alpha|} (\pi_\alpha \otimes \pi'_{\bar{\alpha}^*} + \pi'_{\bar{\alpha}^*} \otimes \pi_\alpha),$$

where $\bar{\alpha}^*$ is the conjugate partition of the complementary partition $\alpha^* = (j - a_i, \dots, j - a_1)$ and π_α (respectively, π'_β) is the Schur polynomial in the first i variables (respectively, last j variables). The conjugate (dual) partition of the partition $\beta = (\beta_1, \dots, \beta_k)$ is the partition $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2, \dots)$, where $\bar{\beta}_j = \#\{i | \beta_i \geq j\}$. We also define $|\alpha| = \sum_{l=1}^i \alpha_l$.

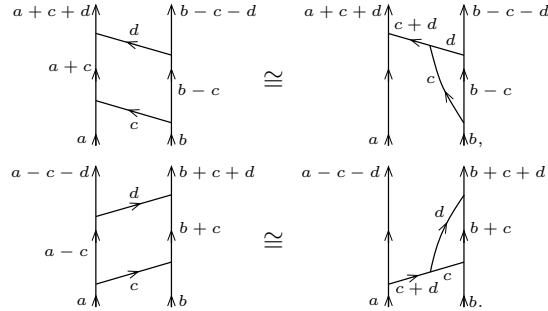
The following lemma is proved in [11]:

Lemma 8.2. μ_{ij} and Δ_{ij} are R_{ij} -bimodule maps.

Note that the lemma above implies that Δ_{ij} is the coproduct in the commutative Frobenius extension $H_{U_{(i+j)}}^*(G(i, i + j))$ with respect to the trace defined by $\text{tr}(\pi_{j \dots j}) = 1$.

The digon creation and annihilation maps are deduced from Lemma 3.1.

Before we explain the complexes that we associate to the crossings, we call attention to the following isomorphisms:



To each positive crossing such that $i \leq j$ we associate a complex of resolutions (see Figure 16)

$$\widehat{\Gamma}_0\{i(i+1)\} \xrightarrow{d_0^+} \widehat{\Gamma}_1\{(i-1)i\} \xrightarrow{d_1^+} \widehat{\Gamma}_2\{(i-2)(i-1)\} \xrightarrow{d_2^+} \dots \xrightarrow{d_{i-1}^+} \widehat{\Gamma}_i,$$

where we put $\widehat{\Gamma}_i$ in the homological degree 0. Our candidates for the differentials are indicated in Figure 17. Note that we have used the isomorphisms above.

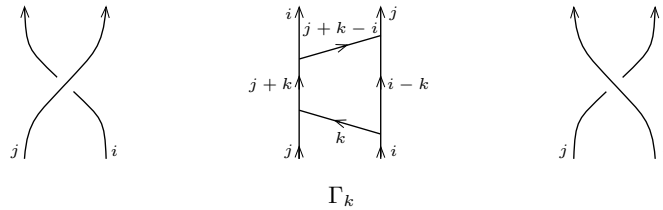


FIGURE 16. A positive crossing, the web Γ_k , and a negative crossing

To negative crossings with $i \leq j$ we associate the following complex of bimodules:

$$\widehat{\Gamma}_i\{-2ij\} \xrightarrow{d_0^+} \widehat{\Gamma}_{i-1}\{-2ij\} \xrightarrow{d_1^+} \widehat{\Gamma}_{i-2}\{2-2ij\} \xrightarrow{d_2^+} \dots \xrightarrow{d_{i-1}^+} \widehat{\Gamma}_0\{(i-1)i-2ij\},$$

where again we put the resolution $\widehat{\Gamma}_i$ in the homological degree 0. The differentials are obtained by inverting all arrows in Figure 17.

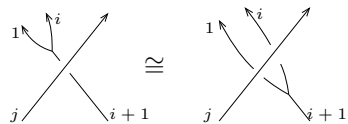
In the case when $i < j$, by looking at the picture from the “other side of the paper” (i.e. by rotation around the y -axis), we obtain the crossing of the above form.

The following lemma is hard to prove directly, as one has to check the statement for a lot of generators. However, in a subsequent paper we will prove this conjecture using a different technique.

Conjecture 8.3. $d_{k+1}^\pm d_k^\pm = 0$ for all $0 \leq k \leq i - 2$.

The other tricks remain largely the same. The crucial conjecture to prove is

Conjecture 8.4. We have the homotopy equivalence



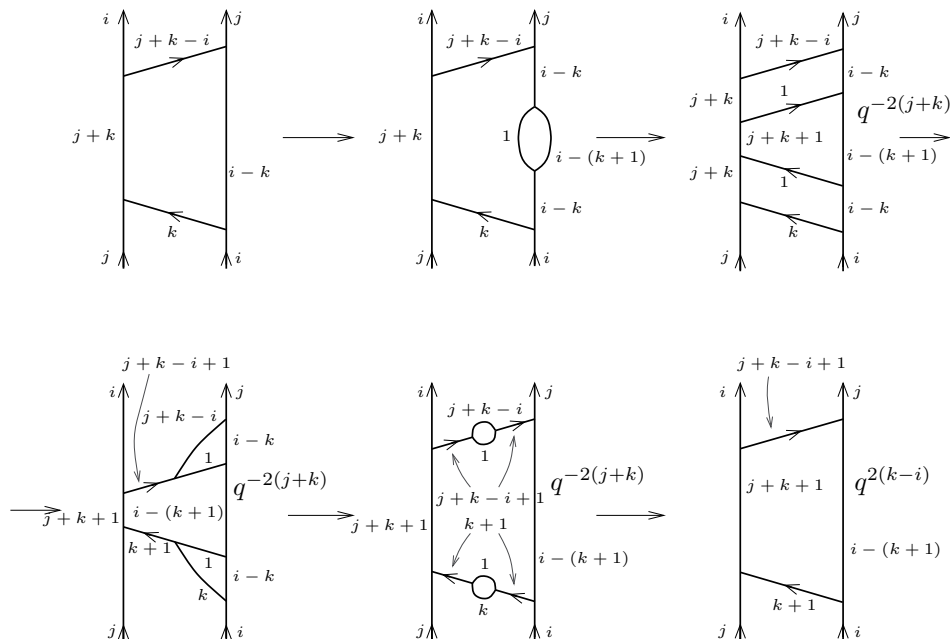


FIGURE 17. The definition of the differentials

In the meantime Webster and Williamson [14] have obtained the same complex for the coloured HOMFLY-PT link homology as we propose in this section, using geometric methods. This proves Conjecture 8.3, but it would still be nice to have an algebraic proof of this result as well. They have also proved invariance. Since their method is different, it is not clear to us whether it proves our Conjecture 8.4. Anyway, it would be interesting to have an algebraic proof of that conjecture.

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